ANALYTIC CONTINUATION OF HOLOMORPHIC MAPPINGS FROM NONMINIMAL HYPERSURFACES

I. KOSSOVSKIY AND R. SHAFIKOV

ABSTRACT. We study the analytic continuation problem for a germ of a biholomorphic mapping from a nonminimal real hypersurface $M \subset \mathbb{C}^n$ into a real hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ and prove that under certain nondegeneracy conditions any such germ extends locally biholomorphically along any path lying in the complement $U \setminus X$ of the complex hypersurface X contained in M for an appropriate neighbourhood $U \supset X$. Using the monodromy representation for the multiple-valued mapping obtained by the analytic continuation we establish a connection between nonminimal real hypersurfaces and singular complex ODEs.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let $H(\zeta,\bar\zeta)$ be a nondegenerate Hermitian form in $\mathbb{C}^{n+1}$ with $k+1$ positive and $l+1$ negative eigenvalues, $k+l=n-1$, $0\le l\le k\le n-1$. We call a hypersurface $\mathcal{Q}\subset\mathbb{CP}^n$ a (k,l)-*hyperquadric* if it is given in homogeneous coordinates by

$$\mathcal{Q}=\left\{[\zeta_0,\dots,\zeta_n]\in\mathbb{CP}^n : H(\zeta,\bar\zeta)=0\right\}. \tag{1}$$

Clearly, $\mathcal{Q}\subset\mathbb{CP}^n$ is a compact smooth real algebraic Levi nondegenerate hypersurface with (k,l) being the signature of its Levi form. In particular, the unit sphere $S^{2n-1}\subset\mathbb{C}^n$ is an $(n-1,0)$-hyperquadric.

Let M be a connected smooth real analytic hypersurface in $\mathbb{C}^n$, $n>1$. It was shown in [19] for $\mathcal{Q}=S^{2n-1}$ and in [12] for the general case that if M is Levi nondegenerate then a

Date: March 21, 2012.

2000 *Mathematics Subject Classification.* 32D15, 32V40, 32H02, 32H04, 32H35, 32M99, 32T25, 34M35.

Key words and phrases. Analytic continuation, holomorphic mappings, real hypersurfaces, nonminimal hypersurfaces, Segre varieties, Levi nondegeneracy, automorphism groups.

germ of a local biholomorphic map $f : M \to \mathcal{Q}$ extends locally biholomorphically along any path on M with the extension sending M to $\mathcal{Q}$. This leads to the following definition: a Levi nondegenerate hypersurface M is called (k,l)-*spherical* at a point $p \in M$ if there exists a germ at p of a biholomorphic map f sending the germ (M,p) of M at p onto the germ of a (k,l)-hyperquadric $\mathcal{Q}$ at $f(p)$. It follows then that M is (k,l)-spherical at one point iff it has this property at all points and we simply call M a (k,l)-*spherical* hypersurface. Similar extension result holds if instead of Levi nondegeneracy one assumes that M is *essentially finite*, a condition on the so-called Segre map of M generalizing Levi nondegeneracy, see [24] and [12]. Further, analytic continuation also holds in the case when $\mathcal{Q} = S^{2n-1}$ and M is *minimal*, i.e., when M does not contain any germs of complex hypersurfaces, see [22].

In this paper we study the analytic continuation phenomenon for biholomorphic maps from a *nonminimal* hypersurface M, i.e., when M contains a complex hypersurface X. In this case the Levi form of M vanishes identically on X, and M is not essentially finite at points in X. For $n = 2$ nonminimality is equivalent to the *infinite type* condition (see [1]). The following illuminating example shows that the above analytic continuation phenomenon fails in general in the nonminimal case.

Example 1.1. (V. Beloshapka, A. Loboda, 2001, see also [4], [15]). Consider the real-analytic hypersurface given by

$$M^{\log} = \left\{(z, w = u + iv) \in \mathbb{C}^2 : v = u \, \tan |z|^2, \ |z| < 1\right\}, \tag{2}$$

or by a global "complex defining equation" $w = \bar{w}e^{2iz\bar{z}}$. Note that $M^{\log}$ contains the complex hypersurface $X = \{w = 0\}$, but it is Levi nondegenerate at all other points. The set X divides $M^{\log}$ into two connected components: M^+ given by $\{u > 0\}$ and M^- given by $\{u < 0\}$. It follows that

$$M \setminus X = \left\{\arctan \frac{v}{u} = |z|^2\right\},$$

and so for $u > 0$, we have $\mathrm{Im}\,(\ln w) = |z^2|$. This shows that the map $F : \{u > 0\} \to \mathbb{C}^2$ given by

$$z^* = z, \, w^* = \ln w, \quad -\frac{\pi}{2} < \mathrm{Arg}\, w < \frac{\pi}{2},$$

maps M^+ onto an open subset of the nondegenerate hyperquadric

$$\mathcal{Q} = \{(z^*, w^*) \in \mathbb{C}^2 : \mathrm{Im}\, w^* = |z^*|^2\}.$$

However, F clearly does not extend across X (neither holomorphically nor as a holomorphic correspondence). In fact, the branch $F^- : \{u < 0\} \to \mathbb{C}^2$ of the multiple-valued map $z^* = z, \, w^* = \ln w$ satisfying $\frac{\pi}{2} < \mathrm{Arg}\, w < \frac{3\pi}{2}$ sends M^- into an open subset of a *different* hyperquadric

$$\tilde{\mathcal{Q}} = \{(z^*, w^*) \in \mathbb{C}^2 : \mathrm{Im}\, w^* - \pi = |z^*|^2\}.$$

This model example and many more examples given later in this paper suggest that given a nonminimal hypersurface M containing a complex hypersurface X, and a local map f from M into a hyperquadric $\mathcal{Q}$, one cannot expect in general that f extends holomorphically to X, and since the complement of X is not simply connected, the analytic continuation, if exists, can lead to a multiple-valued extension of f. Furthermore, since $M \setminus X$ is not connected, different components of $M \setminus X$ can be mapped by different branches of the multiple-valued extension into different hyperquadrics.

Our principal result establishes such multiple-valued holomorphic extension phenomenon. We call a real hypersurface M containing a complex hypersurface X *pseudospherical* if at least one of the components of $M \setminus X$ is (k,l)-spherical for some $k + l = n - 1$, $0 \le l \le k \le n - 1$.

Theorem 1. *Let $M \subset \mathbb{C}^n$ be a connected smooth real-analytic hypersurface containing a complex hypersurface X. Assume that $M \setminus X$ is Levi nondegenerate, and that M is pseudospherical. Then:*

(i) There exists a neighbourhood U of X such that for $p \in (M \setminus X) \cap U$ any biholomorphic map f of (M, p) into a (k, l) - hyperquadric $\mathcal{Q}$ extends analytically along any path in $U \setminus X$ as a locally biholomorphic map into $\mathbb{CP}^n$. In particular, f extends to a possibly multiple-valued locally biholomorphic analytic mapping $U \setminus X \longrightarrow \mathbb{CP}^n$ in the sense of Weierstrass.

(ii) If one of the components of $M \setminus X$ is (k, l) - spherical, then the second component is (k', l') - spherical with, possibly, $(k, l) \neq (k', l')$.

Somewhat surprisingly, the result in (ii) cannot be strengthened: in Example 6.2 of Section 6 we construct a real hypersurface $M \subset \mathbb{C}^3$ for which $(k, l) \neq (k', l')$. However, if f extends to $U \setminus X$ as a single-valued map, then both components of $M \setminus X$ have the same signature of the Levi form as shown in Proposition 6.4.

The nature of multiplicity of the extension in the above theorem depends only on the geometry of the hypersurface M, and does not depend on the choice of the map f. In Section 7 we give a precise description of the monodromy of analytic continuation of f about X by constructing explicitly the monodromy operator, in analogy with the corresponding theory of singular ODEs. To suppress technical details we give a simplified formulation of our result below and refer the reader to Section 7 for further details.

Theorem 2. *Under the conditions of Theorem 1, there exists a linear representation*

$$\phi : \pi_1(U \setminus X) \longrightarrow \mathrm{Aut}(\mathbb{CP}^n)$$

such that the analytic continuation $\tilde{f}$ of f along a cycle $\gamma \subset U \setminus X$ satisfies

$$\tilde{f} = \phi(\gamma) \circ f.$$

The cyclic subgroup $\phi(\pi_1(U \setminus X)) \subset \mathrm{Aut}(\mathbb{CP}^n)$ is determined by M uniquely up to conjugation.

As another application of Theorem 1 we obtain in Section 9 some results concerning the representation of local automorphisms of M. In particular, some further progress can be made in the direction of the so-called Dimension Conjecture. The main result can be formulated as follows.

Theorem 3. *Under the conditions of Theorem 1, suppose $0 \in X \subset M$. Then the (pseudo)group $\mathrm{Aut}\,(M, 0)$ of local automorphisms of M at the origin admits a natural embedding*

$$\mathrm{Aut}\,(M, 0) \longrightarrow Z(\sigma) \cap \mathrm{Aut}(\mathcal{Q}) \subset \mathrm{Aut}(\mathbb{CP}^n),$$

where $Z(\sigma)$ is the centralizer in $\mathrm{Aut}(\mathbb{CP}^n)$ of the element σ, generating the above cyclic subgroup $\phi(\pi_1(U \setminus X))$.

The paper is organized as follows. In Section 2 we provide the necessary information about Segre varieties and the associated notion of Segre map, and prove the local one-to-one property of the Segre map in $U \setminus X$ for hypersurfaces under consideration. In Section 3 we use this property to show that *any* point in the punctured neighbourhood $U \setminus X$ can be connected with $q \in M \setminus X$ by a chain of Segre varieties. We use this fact to construct in Section 4 the desired analytic continuation along some specific paths by means of extension along Segre varieties, in the spirit of [24], [12], and [8]. We also introduce the notion of $\mathcal{Q}$-Segre property for a map f and use it for appropriate understanding of the extension along (iterated) Segre varieties. Combining the results of Sections 3 and 4, we prove a crucial corollary claiming that the initial germ F_0 can be extended to an *arbitrary* point $r \in U \setminus X$ along some specific path. In Section 5 we use this result to prove the continuation along an arbitrary path, using the global nature of the (complexified) automorphism group of the

target hyperquadric $\mathcal{Q}$, which gives the first part of our principal result. We also provide a number of examples of (multiple-valued) analytic maps that extend germs of local biholomorphic mappings to a hyperquadric. Most of the examples are certain blow-ups of the unit 3-sphere (both single-valued and multiple-valued examples occur). In Section 6 we prove the second part of Theorem 1 and give an example of M that can be mapped to *inequivalent* hyperquadrics. In Section 7 we describe the monodromy of the obtained multiple-valued map showing that the monodromy can be expressed in terms of a (scaled) element of $\mathrm{GL}_{n+1}(\mathbb{C})$ - the monodromy matrix. We also establish an intriguing connection between nonminimal pseudospherical hypersurfaces and linear differential equations of order n with an isolated singular point by proving the *Monodromy formula* for the multiple-valued mapping F. The hypersurface $X \subset M$, playing the role of an isolated singularity for holomorphic maps under consideration, becomes an analogue of a single point in $\mathbb{CP}^1$ as an isolated singularity of linear differential equations. In Section 8 we consider separately the case where M is algebraic and prove that the multiple-valued mapping F in this case extends to X, either holomorphically or as a holomorphic correspondence. Finally, in Section 9 we apply the linear monodromy principle from Section 7 to obtain an interesting representation of the local holomorphic automorphism group of M as a subgroup in the centralizer of the monodromy matrix. Then we demonstrate the essential progress that this representation gives to the so-called Dimension Conjecture.

Acknowledgments. We would like to thank V. Beloshapka and B. Lamel for fruitful discussions that motivated in many aspects the current research project. The second author is partially supported by the Natural Sciences and Engineering Research Council of Canada.

2. Background: Segre varieties

Let M be a smooth real analytic hypersurface in $\mathbb{C}^n$, $n \geq 2$, $0 \in M$, and U a neighbourhood of the origin. In what follows in this paper we consider only connected real hypersurfaces. If U is sufficiently small then $M \cap U$ admits a real analytic defining function $\rho(Z, \overline{Z})$, and for every point $\zeta \in U$ we can associate to M its so-called Segre variety in U defined as

$$Q_\zeta = \{Z \in U : \rho(Z, \overline{\zeta}) = 0\}. \tag{3}$$

Note that Segre varieties depend holomorphically on the variable $\overline{\zeta}$. In fact, we may find a suitable pair of neighbourhoods $U_2 = U_2^z \times U_2^w \subset \mathbb{C}^{n-1} \times \mathbb{C}$ and $U_1 \Subset U_2$ such that

$$Q_\zeta = \left\{(z, w) \in U_2^z \times U_2^w : w = h(z, \overline{\zeta})\right\}, \quad \zeta \in U_1, \tag{4}$$

is a closed complex analytic subset. Here h is a holomorphic function. Following [8] we call U_1, U_2 a *standard pair of neighbourhoods* of the origin. From the definition and the reality condition on the defining function the following basic properties of Segre varieties easily follow:

$$Z \in Q_\zeta \;\Leftrightarrow\; \zeta \in Q_Z, \tag{5}$$

$$Z \in Q_Z \;\Leftrightarrow\; Z \in M, \tag{6}$$

$$\zeta \in M \Leftrightarrow \{Z \in U_1 : Q_\zeta = Q_Z\} \subset M. \tag{7}$$

The set $I_\zeta := \{Z \in U_1 : Q_\zeta = Q_Z\}$ is also a complex analytic subset of U_1. If M contains a complex hypersurface X, then for any $p \in X$, $Q_p = X$ and $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X$, so $I_p = X$.

If $f : U \to U'$ is a holomorphic map sending a smooth real analytic hypersurface $M \subset U$ into another such hypersurface $M' \subset U'$, and U is as in (4), then

$$f(Z) = Z' \quad\Longrightarrow\quad f(Q_Z) \subset Q'_{Z'}, \tag{8}$$

for Z close to the origin. The invariance property of Segre varieties will play a fundamental role in our arguments. For the proofs of these and other properties of Segre varieties see, e.g., [7], [8], [9], or [1].

The space of Segre varieties $\{Q_Z : Z \in U_1\}$ can be identified with a subset of $\mathbb{C}^N$ for some $N > 0$ in such a way that the so-called Segre map $\lambda : Z \to Q_Z$ is holomorphic (see [7]). Since we have $Q_p = X$ for any $p \in X$, the Segre map λ sends the entire X to a unique point in $\mathbb{C}^N$ and, accordingly, λ is not even finite-to-one near each $p \in X$ (i.e., M is *not essentially finite* at points $p \in X$). On the other hand, if M is Levi nondegenerate at a point p, then its Segre map is one-to-one in a neighbourhood of p. In fact, the last property can be strengthened as follows.

Proposition 2.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, containing a complex hypersurface X, $0 \in X \subset M$. Suppose that $M \setminus X$ is Levi nondegenerate. Then the standard pair of neighbourhoods (U_1, U_2) for $0 \in M$ can be chosen in such a way that the Segre map $\lambda : U_1 \longrightarrow \mathbb{C}^N$ is locally injective at any point $p \in U_1 \setminus X$.*

Proof. Denote by Σ the set of points where the rank of the map λ is less than n. Clearly, Σ is a complex-analytic subset of U_1, and $X \subset \Sigma$. We will show that U_1 can be taken sufficiently small so that $\Sigma \cap U_1 = X \cap U_1$. Let $\tilde{\Sigma}$ be any irreducible component of Σ of positive dimension such that $0 \in \tilde{\Sigma}$. It follows from injectivity of λ at Levi nondegenerate points that $\tilde{\Sigma} \cap M \subset X$.

Let U^+ and U^- be the two connected components of $U_1 \setminus M$. We claim that either $\tilde{\Sigma} \subset \overline{U^+}$ or $\tilde{\Sigma} \subset \overline{U^-}$. Indeed, suppose that on the contrary, $\tilde{\Sigma} \cap U^+ \neq \emptyset$ and $\tilde{\Sigma} \cap U^- \neq \emptyset$. Let $d := \dim \tilde{\Sigma}$. First observe that $\tilde{\Sigma} \cap M \not\subset \tilde{\Sigma}^{\mathrm{sing}}$, for otherwise the set $\tilde{\Sigma}^{\mathrm{sing}}$ would divide $\tilde{\Sigma}^{\mathrm{reg}}$ into a union of two open components (because M divides U_1, and therefore $\tilde{\Sigma} \cap M$ divides $\tilde{\Sigma}$). This is however impossible because for irreducible $\tilde{\Sigma}$, the set $\tilde{\Sigma}^{\mathrm{reg}}$ is connected (see, e.g., [6]). It follows that $\tilde{\Sigma} \cap M$ contains regular points of $\tilde{\Sigma}$, and, considering a small neighbourhood of any such point, we conclude that the dimension of the real-analytic set $\tilde{\Sigma} \cap M$ equals $2d - 1$ (since this set splits $\tilde{\Sigma}^{\mathrm{reg}}$). On the other hand, $\tilde{\Sigma} \cap X \subset \tilde{\Sigma} \cap M \subset \tilde{\Sigma} \cap X$ from the above arguments, so $\tilde{\Sigma} \cap M = \tilde{\Sigma} \cap X$, which shows that the dimension of $\tilde{\Sigma} \cap M$ can not be odd. That proves the claim.

Now if, for example, $\tilde{\Sigma} \subset \overline{U^+}$, we can move $\tilde{\Sigma}$ along the normal direction to M at 0 and get $\tilde{\Sigma} \cap W \subset U^+$ for the perturbed set $\tilde{\Sigma}$ and a sufficiently small neighbourhood W of the origin. This means that $\tilde{\Sigma} \cap X \neq \emptyset$, though for the perturbed set $\tilde{\Sigma} \cap X = \emptyset$, which is a contradiction, because $\dim \tilde{\Sigma} + \dim X \geq n$ and therefore their intersection is stable under small perturbations ([6]).

From the above we conclude that all components of Σ, different from X, do not intersect X. The zero-dimensional components of Σ do not accumulate at 0, and therefore, we may choose the neighbourhood U_1 so small that $\Sigma = X \cap U_1$, as required. ☐

Proposition 2.1 motivates the following

Definition 2.2. A smooth real-analytic hypersurface M, containing a complex hypersurface $X \ni 0$, is called *Levi-regular in a neighbourhood U of the origin,* if the Segre map λ of M is locally injective at each point $p \in U \setminus X$.

We immediately conclude from Proposition 2.1, that *for a smooth real-analytic hypersurface M, containing a complex hypersurface $X \ni 0$ and Levi nondegenerate in $U \setminus X$, the standard pair of neighbourhoods U_1, U_2 of the origin can be chosen in such a way that M is Levi-regular in U_1.*

The Levi-regularity will be the basic assumption for most of the statements in this paper. We note, once again, that for a Levi-regular at a neighbourhood U hypersurface the image $\lambda(X)$ consists of a unique point in $\mathbb{C}^N$ and near all points $p \in U \setminus X$ the map λ is one-to-one.

Finally we describe the geometry of Segre varieties for the nondegenerate hyperquadric $\mathcal{Q}$ in the target domain. In this case, the Segre variety of a point $\zeta \in \mathbb{CP}^n$ is the projective hyperplane

$$Q'_\zeta = \{\xi \in \mathbb{CP}^n : H(\xi, \bar{\zeta}) = 0\},$$

and the set $\{Q'_\zeta, \zeta \in \mathbb{CP}^n\}$ of all Segre varieties coincides with the space $(\mathbb{CP}^n)^*$ of all projective hyperplanes in $\mathbb{CP}^n$. The Segre map λ' in this case is a global natural one-to-one correspondence between $\mathbb{CP}^n$ and the space $(\mathbb{CP}^n)^*$.

3. Exhaustion of a punctured neighbourhood by Segre sets

Let M, X, U_1, U_2 be as in Section 2. Following [1], we introduce the *Segre sets* of M in a neighbourhood of the origin. We choose $q \in U_1$ and define the zero and the first Segre sets S_0^q, S_1^q of q simply as $S_0^q := \{q\}$ and $S_1^q := Q_q \cap U_1$. Higher order Segre sets S_j^q, $j \geq 2$ are defined by induction as

$$S_j^q := \left(\bigcup_{r \in S_{j-1}^q} Q_r \right) \cap U_1.$$

Finally, we define $S_\infty^q := \bigcup\limits_{j \geq 0} S_{2j}^q$. For $q \in X$ we have $S_j^q = X \cap U_1$ for any $j \geq 0$. As it is shown in [1], Segre sets have the following properties.

(a) $S_j^q \subset S_{j+2}^q$ for $q \in U_1$ and $S_j^q \subset S_{j+1}^q$ for $q \in M \cap U_1$.
(b) $r \in S_j^q \Leftrightarrow q \in S_j^r$ and so $r \in S_\infty^q \Leftrightarrow q \in S_\infty^r$.
(c) S_j^q can be presented as $\pi(\sigma_j^q)$, where $\sigma_j^q \subset \mathbb{C}^N$ is a complex submanifold ($N > n$), and $\pi : \mathbb{C}^N \longrightarrow \mathbb{C}^n$ is a holomorphic projection.

In this section we show that the open connected set $U_1 \setminus X$ can be exhausted by the even Segre sets S_{2j}^p.

Proposition 3.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, containing a complex hypersurface $X \ni 0$, and U_1, U_2 be the standard pair of neighbourhoods for M at the origin. Suppose that M is Levi regular in U_1. Then for any $q \in U_1 \setminus X$,*

$$S_\infty^q = U_1 \setminus X.$$

Proof. Property (b) above shows that for any two Segre sets S_∞^q, S_∞^r, $q, r \in U_1$ either $S_\infty^q = S_\infty^r$ or $S_\infty^q \cap S_\infty^r = \emptyset$ holds. So $U_1 \setminus X$ can be represented as a disjoint union of some S_∞^q, $q \in U_1 \setminus X$ (since each $q \in S_2^q$). The proposition now asserts that, in fact, this disjoint union consists of just one element S_∞^q.

We first claim that every S_2^q, $q \in U_1 \setminus X$, is open at any point $r \in S_2^q$, sufficiently close to q except, possibly, the point $r = q$. Indeed, let $U(q)$ be a neighbourhood of q such that the Segre map λ is one-to-one in $U(q)$. Take any point $r \in S_2^q$ so that $r \neq q$ and $r \in U(q)$. Then $r \in Q_s$, $s \in Q_q \cap Q_r$, in particular, $Q_r \cap Q_q \neq \emptyset$, and so injectivity of λ in $U(q)$ implies that $Q_r \neq Q_q$. A sufficiently small perturbation of r does not change the properties $Q_r \neq Q_q$ (from the definition of $U(q)$) and $Q_r \cap Q_q \neq \emptyset$ (as in the proof of Proposition 2.1, we use the fact that the sum of the dimensions of these two analytic sets is at least n and refer to [6]). So for any r', sufficiently close to r, there exists a point s' such that $s' \in Q_q$ and $s' \in Q_{r'}$, so that $r' \in Q_{s'}$ and $r' \in S_2^q$, as required. This proves the claim.

Now take any S_∞^q, $q \in U_1 \setminus X$ and consider an interior point $q' \in S_2^q$. Take a ball B, centred at q' and such that $B \subset S_2^q$. Then for all r sufficiently close to q we have $S_2^r \cap B \neq \emptyset$ (this follows

from the continuity of the map $\lambda: z \longrightarrow Q_z$). Therefore, there exists $r' \in B$ such that $r' \in S^r_2$. We get the inclusions $r \in S^2_{r'}$, $r' \in S^2_q$ which imply, by definition of Segre sets, the inclusion $r \in S^q_4$ for all r sufficiently close to q. This shows that q is an interior point of S^q_4.

Taking a point $s \in S^q_{2j}$ for some $j \geq 0$, we use a similar argument to conclude from openness of S^q_4 at q, that s is an interior point of S^q_{2j+4}. This finally shows that all points of the set S^q_∞ are in fact its interior points and so S^q_∞ is an open set. The connectivity of $U_1 \setminus X$ now implies that the decomposition of $U_1 \setminus X$ into Segre sets consists of a unique element and $U_1 \setminus X = S^q_\infty$ for any $q \in U_1 \setminus X$, as required. □

Example 3.2. For the hypersurface $M^{\log}$ (see Introduction) we may choose $U_1 = U_2 = \mathbb{C}^2$ and $p = (0,1) \in M$. Then M is Levi-regular in U_1 and simple computations show:

$$S^p_0 = p,\ S^p_1 = \{w = 1\},\ S^p_2 = (\mathbb{C}^2 \setminus X) \setminus \{z = 0,\ w \neq 1\},\ S^p_3 = S^p_4 = ... = \mathbb{C}^2 \setminus X.$$

It is also not difficult to see, that taking $U_1 = \{|z| < \varepsilon,\ |w| < \varepsilon\}$ and $p = (\frac{\varepsilon}{2}, 0)$, all points, lying in the j-th Segre set S^p_j, satisfy: $|w| \geq \frac{1}{2}\varepsilon e^{-2j\varepsilon^2}$. This inequality shows that no Segre set of a fixed "depth" j can a priori exhaust the punctured neighbourhood $U_1 \setminus X$ for a nonminimal Levi-regular hypersurface M.

4. Extension along Segre varieties

The result of the previous section, showing that iterated Segre varieties of a fixed point $p \in M \setminus X$ exhaust the punctured neighbourhood $U_1 \setminus X$, suggests that the desired continuation of a given local biholomorphic map F of M into a quadric $\mathcal{Q}$ can be obtained by extending F along iterated Segre varieties of the point p. The extension along Segre varieties is based on their invariance property (8) and gives an effective way of holomorphic continuation for holomorphic maps of real submanifolds in complex spaces (see [24], [12]).

Let M, X, U_1, U_2 be as in Section 2, with $0 \in X \subset M$. Let $p \in (M \setminus X) \cap U_1$. We first introduce the following notation: by $Q_{p_0,p_1,...p_{j-1}}$ we denote the Segre variety $Q_{p_{j-1}}$, where $p_0 := p$, $p_k \in Q_{p_{k-1}}$, $k = 1, 2, ..., j-1$ so that $p_k \in S^p_k$, $k = 0, 1, 2, ..., j-1$ and $Q_{p_0,p_1,...p_{j-1}} \subset S^p_j$. In this section we show that a local biholomorphic map F of M into a hyperquadric can be extended, in a certain sense, to a neighbourhood of any $Q_{p_0,p_1,...p_{j-1}}$. For $j = 1$ the Segre variety Q_{p_0} contains p and the extension can be understood naturally, while for $j \geq 2$ the meaning of extension will be specified.

Let $r \in U_1 \setminus X$, $U(r) \subset U_1 \setminus X$ be an open polydisc, centred at r, and $F: U(r) \longrightarrow \mathbb{CP}^n$ be a biholomorphic map onto its image. For $q \in U(r)$ and $s \in Q_q$ so that $q \in Q_s$, we denote by $(Q_s)^c$ the connected component of $Q_s \cap U(r)$, containing q. We say that F *has the $\mathcal{Q}$-Segre property in* $U(r)$, if for any $q \in U(r)$ and $s \in Q_q$ the image $F((Q_s)^c)$ is contained in a projective hyperplane in $\mathbb{CP}^n$ (which is a Segre variety of $\mathcal{Q}$).

We now formulate the key claim for the main result of this paper.

Proposition 4.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, $X \subset \mathbb{C}^n$ be a complex hypersurface, $0 \in X \subset M$, (U_1, U_2) be a standard pair of neighbourhoods at the origin. Suppose that M is Levi-regular in U_1 and that for some point $p \in M \setminus X$ and a polydisc $U(p)$, centred at p, there exists a biholomorphic map $F_0: U(p) \longrightarrow \mathbb{CP}^n$ such that $F_0(M \cap U(p)) \subset \mathcal{Q}$, where $\mathcal{Q} \subset \mathbb{CP}^n$ is a nondegenerate real hyperquadric. Then there exist connected neighbourhoods $W_1, W_2, ...W_j$ of $Q_{p_0}, Q_{p_1}, ..., Q_{p_{j-1}}$ respectively (so that $p_k \in W_k$, $k = 0, 1, ..., j-1$) and locally biholomorphic maps*

$$F_1, F_2, ..., F_j, \quad F_k: W_k \longrightarrow \mathbb{CP}^n, \quad k = 1, 2, ..., j,$$

such that:

(i) *The intersection* $U(p) \cap W_1$ *contains a polydisc* W_0*, centred at* p_0*, such that* F_1 *is a holomorphic extension of* $F_0|_{W_0}$.
(ii) *For each* $k = 2, .., j$ *the intersection* $W_{k-2} \cap W_k$ *contains a polydisc* $U(p_{k-2})$*, centred at* p_{k-2}*, such that* F_k *is a holomorphic extension of* $F_{k-2}|_{U(p_{k-2})}$.
(iii) *For each* $r \in Q_{p_k}$, $k = 0, 1, ..., j-1$ *there exists a polydisc* $U(r) \subset W_{k+1}$ *such that* $F_{k+1}|_{U(r)}$ *has the* $\mathcal{Q}$*-Segre property in* $U(r)$.

Proof. We use the coordinate system in the preimage in the form $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and denote by U^z and U^w the projections of a polydisc $U \subset U_1 \subset \mathbb{C}^n$ onto the z-coordinate subspace and the w-axis respectively. We also suppose that in these coordinates Segre varieties of M are graphs of the form $w = h(z)$, $h \in \mathcal{O}(U^z)$ and X is given by $\{w = 0\}$. In the target domain we denote by Q'_ζ the Segre varieties of points $\zeta \in \mathbb{CP}^n$ with respect to the hyperquadric $\mathcal{Q}$.

Step 1. We first prove part (i) and (iii) for $k = 0$. We choose $W_0 \subset U(p)$ to be a polydisc, centred at p and such that for each Segre variety Q_q, $q \in Q_p \cap U_1$ the intersection $Q_q \cap W_0$ is the graph of a function over W_0^z, in particular, it is connected (the existence of such a polydisc follows from the fact that $p \in Q_p$, $p \in Q_q$ for $q \in Q_p$ and that the 1-jets of Segre varieties Q_q, $q \in Q_p$ at p are bounded in the intersection of Q_p with the closed polydisc $\overline{U_1} \subset U_2$). So we can choose a connected neighbourhood $W_1 \supset Q_p$ such that for $s \in W_1$ the intersection $Q_s \cap W_0$ is also connected and nonempty.

We follow the strategy in [8] and [24], and consider the set

$$A_1 = \{(Z, \zeta) \in W_1 \times \mathbb{CP}^n : F_0(Q_Z \cap W_0) \subset Q'_\zeta\}.$$

In the same way as it is done in Proposition 3.1 in [24], one can show that A_1 is a nonempty closed complex-analytic subset in $W_1 \times \mathbb{CP}^n$ of dimension n. But unlike the situation in [24], we do not need to exclude an analytically constructible set from Q_p since the hypersurface in the target domain is the hyperquadric $\mathcal{Q}$ whose Segre map is globally injective. If for some $Z \in W_1$, $\zeta_1, \zeta_2 \in \mathbb{CP}^n$ and $\zeta_1 \neq \zeta_2$, then

$$(Z, \zeta_1), (Z, \zeta_2) \in A_1 \implies F_0(Q_Z \cap W_0) \subset Q_{\zeta_1} \cap Q_{\zeta_2},$$

which is not possible since F_0 is biholomorphic in W_0, $\dim Q_{\zeta_1} \cap Q_{\zeta_2} = n - 2$ while $Q_Z \cap W_0$ is of dimension $n - 1$. So A_1 is, in fact, the graph of a holomorphic map $F_1 : W_1 \longrightarrow \mathbb{CP}^n$. To show that F_1 is locally biholomorphic we observe that local injectivity of λ implies that for distinct Z_1, Z_2 that are close to each other, the intersection $Q_{Z_1} \cap Q_{Z_2} \cap W_0$ has dimension at most $n-2$, so by shrinking W_1, if necessary, we conclude that $(Z_1, \zeta) \in A_1$ and $(Z_2, \zeta) \in A_1$ forces $Z_1 = Z_2$, so that F_1 is locally one-to-one and hence biholomorphic. Also, the invariance property of Segre varieties (8) implies that for $Z \in W_1$, sufficiently close to p, $F_0(Z) = F_1(Z)$, which proves (i).

For the proof of (iii) for $k = 0$, we consider the set V_1 of points $q \in U_1$ such that $Q_q \cap W_0 \neq \emptyset$. Clearly,

$$V_1 = \bigcup_{s \in W_0} Q_s. \tag{9}$$

Since W_0 is open, V_1 is also open, and because each Q_s is path-connected and W_0 is open and path-connected, V_1 is also connected. For points $s \in U_1$, close to p, $F_1 = F_0$ and the invariance property implies that F_1 transfers $Q_s \cap W_0$ to an open subset of a projective hyperplane. Now take a point $a \subset W_0^z \subset \mathbb{C}^{n-1}$ and consider an open connected subset $V_a \subset V_1$, which consists of $q \in U_1$ such that $(\{a\} \times W_0^w) \cap Q_q \neq \emptyset$. Clearly, each V_a is open, $V_a = \bigcup_{b \in W_0^w} Q_{(a,b)}$ so that V_a is connected and $V_1 = \bigcup_{a \in W_0^z} V_a$. The set V_a always contains points, sufficiently close to p, and we

may consider on V_a the holomorphic map, which assigns the k-jet, $k \geq 2$ of Q_q, $q \in V_a$ at the point a. This mapping vanishes for points in V_a, close to p (since for such points $F_1(Q_q \cap W_0)$ is contained in a projective hyperplane), and consequently vanishes on entire V_a, so for all $q \in V_a$, F_1 transfers the connected component of $Q_q \cap W_0$, containing the point with z-coordinates equal to a, to an open subset of a projective hyperplane. From this and (9) it follows that the desired $\mathcal{Q}$-Segre property holds for F_1 in W_0.

Now we take any $r \in Q_p$ and prove the $\mathcal{Q}$-Segre property for F_1 is some polydisc $U(r)$. We choose $U(r) \subset W_1$ and so that $Q_p \cap U(r)$ is the graph of a function over $U^z(r)$, so that for p^*, sufficiently close to p, $Q_p \cap U(r)$ is connected. Since $W_1 \supset Q_p$, for p^*, close to p, $Q_{p^*} \subset W_1$, and the $\mathcal{Q}$-Segre property of F_1 in W_0 implies, by the uniqueness property, that $F_1(Q_{p^*})$ is an open subset of a projective hyperplane, and so does $F_1(Q_{p^*} \cap U(r))$. Now arguments analogous to those used above show the $\mathcal{Q}$-Segre property for F_1 in $U(r)$, which completes Step 1.

Step 2. We now prove (ii) and (iii) for $j = 2$. This will give us the base case for a general induction argument. The proof for this case will be a small modification of the one in the previous step.

From (i), $W_0 \subset W_1$ and $F_0|_{W_0} = F_1|_{W_0}$. We choose a polydisc $U(p_1) \subset W_1$ with the $\mathcal{Q}$-Segre property for F_1 and a connected neighbourhood W_2 of Q_{p_1} such that for each Segre variety Q_q, $q \in W_2$ the intersection $Q_q \cap U(p_1)$ is the graph of a function over $U^z(p_1)$, in particular, it is connected. Consider the set

$$A_2 = \{(Z, \zeta) \in W_2 \times \mathbb{CP}^n : F_1(Q_Z \cap U(p_1)) \subset Q_\zeta\}.$$

The $\mathcal{Q}$-Segre property of F_1 and arguments, similar to those in [24] show that A_2 is an nonempty complex analytic set in $W_1 \times \mathbb{CP}^n$ of dimension n. By shrinking W_2 if needed, we obtain in a similar fashion that A_2 defines a locally biholomorphic mapping $F_2 : W_2 \longrightarrow \mathbb{CP}^n$. Since $Q_{p_1} \ni p$, we conclude that $p \in W_2$ and for points $p^* \in W_2$, sufficiently close to p, $Q_{p_2} \subset W_1$ and the intersection $W_0 \cap W_{p^*}$ is connected. By the invariance property of $F_1 = F_0$ in W_0 we conclude that the point in A_2 over p^* must equal $F_1(p^*)$, i.e., $F_2(p^*) = F_1(p^*) = F_0(p^*)$, which proves (ii) for $j = 2$. The proof of (iii) for this case follows the same pattern as in Step 1.

Step 3. We now perform the induction step by assuming that $j > 2$ and that for all smaller j the proposition holds, i.e., for all $k < j$ the desired extensions and polydiscs with the $\mathcal{Q}$-Segre property have been already obtained. We treat the case $k = j$.

In the same way as in Step 2, we obtain, using the $\mathcal{Q}$-Segre property of F_{j-1}, a polydisc $U(p_{j-1})$, a neighbourhood W_j of $Q_{p_{j-1}}$ with $Q_q \cap U(p_{j-1})$ connected for $q \in W_j$ and a locally biholomorphic map $F_j : W_j \longrightarrow \mathbb{CP}^n$, corresponding to the n-dimensional complex analytic set

$$A_j = \{(Z, \zeta) \in W_j \times \mathbb{CP}^n : F_{j-1}(Q_Z \cap U(p_{j-1})) \subset Q_\zeta\}.$$

To prove (ii), take now Z close to p_{j-2} so that Q_Z is contained in W_{j-1}. To clarify what $F_j(Z)$ equals, recall that by assumption the proposition is proved for smaller j. Therefore,

$$F_{j-1}(Q_Z \cap U(p_{j-1})) = F_{j-3}(Q_Z \cap U(p_{j-1})) = Q'_{F_{j-3}(Z)},$$

so by the definition of F_j and F_{j-2} we obtain that $F_j(Z) = F_{j-2}(Z)$, and (ii) is proved.

Finally, to prove (iii) for F_j we take $r \in Q_{p_{j-1}}$ and a polydisc $U(r)$ such that for p^* close to p_{j-1} the intersection $Q_{p^*} \cap U(r)$ is connected. We also may suppose that $Q_{p^*} \subset W_j$. Since $F_{j-2}(Q_{p^*} \cap U(p_{j-2}))$ contained in a projective hyperplane and $F_{j-2} = F_j$ in $U(p_{j-2}) \subset W_j$, we conclude that $F_j(Q_{p^*})$ is contained in a projective hyperplane. To obtain the entire $\mathcal{Q}$-Segre property for F_j we repeat the arguments from the proof in Step 1. This completes the proof of the theorem. $\square$

We now formulate the following corollary, which is a weaker form of our main extension result, but is convenient for applications.

Corollary 4.2. *Let M, X, p, U_1, U_2, F_0 satisfy Proposition 4.1. Then for each point $q \in U_1 \setminus X$ there exists a connected path $\gamma : [0,1] \longrightarrow U_1 \setminus X$, $\gamma(0) = p, \gamma(1) = q$ such that F_0 extends analytically along γ as a locally biholomorphic mapping to $\mathbb{CP}^n$ and for any $r \in \gamma$ there exists a polydisc $U(r)$, centred at r such that the mapping F_r has the $\mathcal{Q}$-Segre property in $U(r)$ (here F_r is the element of the analytically continued germ F_0 along γ at the point r).*

Proof. Proposition 3.1 implies that there exist points $p_1, p_2, ..., p_{2j-1} \in U_1 \setminus X$ such that $q \in Q_{p_0,p_1,...,p_{2j-1}}$, $j \geq 1$. We set $p_{2j} := q$ and choose connected paths $\Gamma_{0,2} \subset Q_{p_1}$, $\Gamma_{2,4} \subset Q_{p_3}, ..., \Gamma_{2j-2,2j} \subset Q_{p_{2j-1}}$ such that

$$\Gamma_{0,2}(0) = p_0,\, \Gamma_{0,2}(1) = p_2,\, \Gamma_{2,4}(0) = p_2,\, \Gamma_{2,4}(1) = p_4, ..., \\ \Gamma_{2j-2,2j}(0) = p_{2j-2},\, \Gamma_{2j-2,2j}(1) = p_{2j} = q. \quad (10)$$

Then, applying Proposition 4.1, we conclude that F_2 is a local biholomorphic extension of F_0 along $\Gamma_{0,2}$; F_4 along $\Gamma_{2,4}$; ... ; F_{2j} is a local biholomorphic extension of F_{2j-2} along $\Gamma_{2j-2,2j}$ (of course, we use the connectivity and simple connectivity of the Segre varieties as holomorphic graphs). Taking now γ to be the union of the paths of $\Gamma_{0,2}, ..., \Gamma_{2j-2,2j}$, we obtain the desired local biholomorphic extension of F_0. The $\mathcal{Q}$-Segre property for F_r, $r \in \gamma$ now follows from Proposition 4.1. □

5. Extension along an arbitrary path

In this section we prove that F_0 can be analytically continued along *any* path in $U_1 \setminus X$. We begin with the following proposition.

Proposition 5.1. *Let M, X, U_1, U_2 satisfy Proposition 2.1, $r \in U_1 \setminus X$, $U(r) \subset U_1 \setminus X$ a polydisc centred at p, and let F, G be two biholomorphic mappings $U(r) \longrightarrow \mathbb{CP}^n$ with $\mathcal{Q}$-Segre property in $U(r)$. Then there exists a linear automorphism τ of $\mathbb{CP}^n$ such that $G = \tau \circ F$.*

Proof. Let $\lambda : U_1 \longrightarrow \mathbb{C}^N$ and $\lambda' : \mathbb{CP}^n \longrightarrow (\mathbb{CP}^n)^*$ be the Segre maps in the preimage and the target domain respectively, $U' = F(U)$. We consider the map $\tau := G \circ F^{-1}$, which is a biholomorphic map $U' \longrightarrow \mathbb{CP}^n$ onto its image.

From the $\mathcal{Q}$-Segre properties of F and G we know that τ maps "many" (connected components of) intersections of projective hyperplanes with U' to open subsets of projective hyperplanes, and want now to prove the same for the set of hyperplanes, intersecting a ball B in some coordinate chart in $\mathbb{CP}^n$. To do so, put $r' := F(r)$ and fix some coordinate ball $B \subset U'$, centred at r'. Let $q \in Q_r$ so that $r \in S_q$. Choose a polydisc $\tilde{U}(r)$ with the following properties:

(i) $\tilde{U}(r) \subset U(r)$, $F(\tilde{U}(r)) \subset B$.

(ii) $S_q \cap \tilde{U}(r)$ is a graph over $\tilde{U}^z(r)$ (we use the notation from Proposition 4.1).

According to property (ii), there exists a connected neighbourhood $V(q)$ such that for each $\tilde{r} \in \tilde{U}(r)$ there exists $\tilde{q} \in V(q)$ such that $\tilde{r} \in Q_{\tilde{q}}$ and $Q_{\tilde{q}} \cap \tilde{U}(r)$ is connected (we simply use the fact that $Q_{\tilde{r}}$ is close to $Q_r \ni q$). Choosing $\tilde{U}(r)$ small enough, we may suppose the Segre map λ is injective in $V(q)$. Consider now the following mapping: taking $\tilde{q} \in V(q)$, we associate $Q_{\tilde{q}}$ to it, then consider $F(Q_{\tilde{q}} \cap \tilde{U}(r))$ - an open subset of a projective hyperplane, and, using λ', associate a point in $(\mathbb{CP}^n)^*$ to it. This is an injective holomorphic map from $V(q)$ to $(\mathbb{CP}^n)^*$, denote its image by W'. Consider also the set of projective hyperplanes, intersecting B, and denote this open connected set in $(\mathbb{CP}^n)^*$ by A'. Then W' is an open subset of A' (by property (i)), and,

by definition of τ, the map τ sends $l_{\mathbf{a}} \cap B$ with $l_{\mathbf{a}} \in W'$ (here $l_{\mathbf{a}}$ is a projective hyperplane that corresponds to $\mathbf{a} \in \mathbb{CP}^n$) to open subsets of projective hyperplanes. Considering, as in the proof of Proposition 4.1, the high order jets of $\tau(l_{\mathbf{a}} \cap B)$ as holomorphic mappings of A', we see that they vanish for $l_{\mathbf{a}} \in W$, so they must vanish for all $l_{\mathbf{a}} \in A'$ and we obtain the desired "hyperplane-to-hyperplane" property of τ for any hyperplane, intersecting B. As it can be verified from many references (see, for example, [26], [25], or [17]) τ in this case must be a local biholomorphic symmetry of the system of flat second order complex differential equations

$$y_{x_k x_l} = 0, \quad k, l = 1, 2, ..., n-1.$$

Hence, it is a linear automorphism of $\mathbb{CP}^n$, and $G = \tau \circ F$, as required. $\square$

We now can prove part (i) of Theorem 1, which we formulate in the following theorem.

Theorem 5.2. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, $X \subset \mathbb{C}^n$ a complex hypersurface, $0 \in X \subset M$. Suppose that M is Levi nondegenerate in $M \setminus X$ and pseudospherical. Then for a suitable neighbourhood U_1 of the origin and a point $p \in (M \setminus X) \cap U_1$, any local biholomorphic mapping $F_0 : (\mathbb{C}^n, p) \longrightarrow \mathbb{CP}^n$, transferring (M, p) onto an open piece of a nondegenerate real hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$, extends analytically along arbitrary continuous path $\gamma : [0,1] \longrightarrow U_1 \setminus X$, $\gamma(0) = p$ as a local biholomorphic mapping into $\mathbb{CP}^n$.*

Proof. Let U_1, U_2 be a standard pair of neighbourhoods of the origin such that M is Levi-regular in U_1. Suppose on the contrary that the claim of the theorem is false. Then, since for t close to 0 the extension with $\mathcal{Q}$-Segre property already exists, we can choose the smallest t_0, $0 < t_0 < 1$, such that F_0 does not extend analytically to $\gamma(t_0)$ along the path $\gamma|_{[0,t_0]}$ with the $\mathcal{Q}$-Segre property in some neighbourhood of each $\gamma(t), 0 \le t \le t_0$ (t_0 is simply the supremum of t such that F_0 extends to $\gamma(t)$ along $\gamma|_{[0,t]}$ with the $\mathcal{Q}$-Segre property at each point). Applying Corollary 4.2, we obtain a polydisc $U(r)$, centred at $r = \gamma(t_0)$ and a mapping $\tilde{F}_r$ in $U(r)$ with the $\mathcal{Q}$-Segre property. We now take some t^* close to t_0 with $t^* < t_0$ and $r^* = \gamma(t^*) \in U(r)$ and denote the corresponding extension of F_0 with the $\mathcal{Q}$-Segre property at some polydisc $U(r^*)$ by F_{r^*}. Without loss of generality we may assume that $U(r^*) \subset U(r)$. Then, applying Proposition 5.1 for the polydisc $U(r^*)$ and mappings $\tilde{F}_r, F_{r^*}$ in it, we get a linear automorphism τ of $\mathbb{CP}^n$ such that $F_{r^*} = \tau \circ \tilde{F}_r$ in $U(r^*)$. This equality clearly implies, by the global nature of τ, that $\tau \circ \tilde{F}_r$ is a holomorphic extension of F_{r^*} to $U(r) \ni r$ with the $\mathcal{Q}$-Segre property, which contradicts the definition of t_0. $\square$

To formulate the following corollary, we will use the set-up of a (multiple-valued) analytic function in the sense of Weierstrass [21] and apply it for mappings $\mathbb{C}^n \longrightarrow \mathbb{CP}^n$. We call the corresponding objects *analytic mappings* in the sense of Weierstrass.

Corollary 5.3. *Let M, X, U_1, F_0, p satisfy Theorem 5.2. Then the mapping $F_0 : (\mathbb{C}^n, p) \longrightarrow \mathbb{CP}^n$ extends locally biholomorphically to a (multiple-valued) analytic mapping $F : U_1 \setminus X \longrightarrow \mathbb{CP}^n$ in the sense of Weierstrass. Moreover, each analytic element (F_r, r) of F at a point $r \in U_1 \setminus X$ has the $\mathcal{Q}$-Segre property.*

Proof. To see the $\mathcal{Q}$-Segre property of (F_r, r), $r \in U_1 \setminus X$ we just follow the proof of Theorem 5.2 and note that $\tau \circ \tilde{F}_r$ is an extension of F_{r^*} with the $\mathcal{Q}$-Segre property. $\square$

Example 5.4. (see also [10],[15],[4]). Consider the standard hyperquadric

$$Q = \{(z^*, w^*) \in \mathbb{C}^2 : \operatorname{Im} w^* = |z^*|^2\}, \tag{11}$$

and the (multiple-valued) locally biholomorphic mappings $F_\alpha : \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{CP}^2$ given as

$$z^* = zw^\alpha,\ w^* = w^{2\alpha},\ \alpha \in \mathbb{R} \setminus \{0\}.$$

Then it is not difficult to check, by plugging F into the defining equation of Q, that F_α^{-1} determined by $-\frac{\pi}{2} < \mathrm{Arg\ w} < \frac{\pi}{2}$ maps (Q, p^*), $p* = (0,1) \in Q$ onto an open piece of the smooth real-analytic hypersurface

$$M_\alpha = \left\{(z,w) \in \mathbb{C}^2 : w = \bar{w}(\sqrt{1-|z|^4} + i|z|^2)^{\frac{1}{\alpha}},\quad |z| < 1\right\}. \tag{12}$$

All M_α are nonminimal, as they contain $X = \{w = 0\}$, and Levi-regular in $|z| < 1$. F_α turns out to be exactly the (multiple-valued) locally biholomorphic mapping, provided by Corollary 5.3. For $\alpha \in \mathbb{Z}$ the mapping F_α is single-valued and extends holomorphically to $X = \{w = 0\}$. Thus F_α^{-1} performs a certain blow-up of the 3-sphere in $\mathbb{C}^2$. For α rational the multiple-valued mapping F_α is finitely-valued and extends to X as a holomorphic correspondence [23] (the graph of F_α extends even to an algebraic subset of $\mathbb{CP}^4$ in this case). For irrational α the mapping F_α is infinitely-valued and, furthermore, the graph of a germ of F_α does not even extend to a closed complex-analytic subset of $(U_1 \setminus X) \times \mathbb{C}^2$ (note that such extension is possible for the model example $M^{\log}$).

Example 5.5. (see also [10]). Considering the quadric Q defined by (11) and the blow-ups G_m given by

$$z^* = zw^m,\ w^* = w,\ m \in \mathbb{Z}, m > 0.$$

The image of Q under G_m gives algebraic hypersurfaces given by

$$K_m = \{\mathrm{Im}\, w = |z|^2 |w|^{2m}\}. \tag{13}$$

These are nonminimal with $X = \{w = 0\}$ and Levi-regular in appropriate polydiscs $U_m(0)$. Here G_m are single-valued and extend to X holomorphically.

6. Application: transfer of sphericity

The above continuation results imply the following remarkable fact on the geometry of nonminimal real hypersurfaces. Throughout the section we denote by M^+ and M^- the connected components of $M \setminus X$. The next theorem is reformulation of Theorem 1, part (ii).

Theorem 6.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, containing a complex hypersurface $X \subset \mathbb{C}^n$ and Levi nondegenerate in $M \setminus X$. Suppose that M is pseudospherical with M^+ being (k,l)-spherical. Then M^- is (k',l') spherical with, possibly, $(k',l') \neq (k,l)$.*

Proof. We fix a standard pair of neighbourhoods U_1, U_2 such that M is Levi-regular in U_1 and choose points $p^+ \in M^+ \cap U_1$ and $p^- \in M^- \cap U_1$ and a local biholomorphic map $F_0 : (\mathbb{C}^n, p^+) \longrightarrow \mathbb{CP}^n$ with $F_+(M^+) \subset \mathcal{Q}$ for a nondegenerate hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ of the signature (k,l). Applying Corollary 4.2, we can find a polydisc $U(p^-)$ and a local biholomorphic map $F_- : U(p^-) \longrightarrow \mathbb{CP}^n$ with $\mathcal{Q}$-Segre property. Put $\mathcal{P} := F_-(M^- \cap U(p^-))$. Then $\mathcal{P} \subset \mathbb{CP}^n$ is a smooth real-analytic Levi nondegenerate hypersurface, biholomorphically equivalent to $M^- \cap U(p^-)$. The $\mathcal{Q}$-Segre property of F_- and the holomorphic invariance of Segre varieties imply that all Segre varieties of $\mathcal{P}$, in some neighbourhood of $F_-(p^-)$, are open pieces of projective hyperplanes. Now choose some affine chart, containing $\mathcal{P}$ and make an invertible affine transformation such that in new coordinates $\mathcal{P}$ has the form

$$2\mathrm{Re}\, w' = H(z', \bar{z}') + O(2),\quad z' \in \mathbb{C}^{n-1}, w \in \mathbb{C},$$

where $H(z',\bar z')$ is a nondegenerate Hermitian form. Then Segre varieties of $\mathcal{P}$ have the form

$$w=-\bar b'+H(z,\bar a')+....$$

This equation determines a hyperplane for all sufficiently small a and b, which implies that all monomials in dots in fact vanish, and therefore, $\mathcal{P}$ is a nondegenerate real hyperquadric. □

The following example shows that, surprisingly, the claim of Theorem 6.1 cannot be strengthened and even algebraic nonminimal hypersurfaces in $\mathbb{C}^n$, $n>2$ may have *different* signature of the Levi form in M^+ and M^-.

Example 6.2. Let $Q=\{\operatorname{Im} w^*=|z_1^*|^2+|z_2^*|^2\}\subset\mathbb{C}^3$ be a real strictly pseudoconvex hyperquadric. Consider the following "blow-up" map F:

$$z_1^*=z_1\sqrt{w},\ z_2^*=z_2w,\ w^*=w.$$

Choosing a connected neighbourhood $D\subset Q$ of the point $(0,0,1)\in Q$ and the single-valued biholomorphic branch of F, given by $-\frac{\pi}{2}<\operatorname{Arg} w^*<\frac{\pi}{2}$, it is straightforward to check that F^{-1} maps D onto an open piece of the smooth real-analytic nonminimal hypersurface

$$M=\left\{w=\bar w\frac{(i|z_1|^2+\sqrt{1-2i|z_2|^2\bar w-|z_1|^4})^2}{(1-2i|z_2|^2\bar w)^2}\right\},$$

satisfying $\operatorname{Re} w>0$ and z_1,z_2,w be small enough (one should rewrite the equation of Q in the new coordinates). It is easy to check that M is Levi nondegenerate outside $w=0$, so M satisfies the conditions of Theorem 6.1, and at the point $p^+\in M^+$, $p^+=(0,0,\varepsilon),\varepsilon>0$ the Levi form is positive definite, though at the point $p^-\in M^-$, $p^-=(0,0,-\varepsilon),\varepsilon>0$ the Levi form has eigenvalues of different signs. So M^+ is $(2,0)$-spherical, though M^- is $(1,1)$-spherical. In fact, the single-valued biholomorphic branch of F given by $\frac{\pi}{2}<\operatorname{Arg}\mathrm{w}<\frac{3\pi}{2}$ maps the negative half M^- of M onto a domain on the indefinite hyperquadric $Q^-=\{\operatorname{Im} w^*=-|z_1^*|^2+|z_2^*|^2\}$.

Unlike the case $n\geq 3$, for $n=2$ all hyperquadrics in $\mathbb{CP}^2$ are equivalent to the 3-sphere $S^3\subset\mathbb{C}^2$ and the phenomenon from Example 6.2 can not hold. However, it may still happen that the multiple-valued mapping, obtained in Theorem 5.2, maps M^+ and M^- to *different* hyperquadrics in $\mathbb{CP}^2$, though these hyperquadrics are equivalent by means of some $\tau\in\operatorname{Aut}(\mathbb{CP}^2)$.

Example 6.3. Consider the hypersurface $M^{\log}\subset\mathbb{C}^2$ (see Introduction). Then the multiple-valued map $F:(z,w)\longrightarrow(z,\ln w)$ maps the domain $M^+\subset M$, given by the condition $u>0$, to the hyperquadrics

$$\{\operatorname{Im} w^*+2k\pi=|z^*|^2\},\quad k\in\mathbb{Z},$$

and the domain $M^-\subset M$, given by $u<0$, to the hyperquadrics

$$\{\operatorname{Im} w^*+(2k+1)\pi=|z^*|^2\},\quad k\in\mathbb{Z}.$$

Each of the hyperquadrics looks as $\tau^k(\mathcal{Q}_0)$, where $\mathcal{Q}_0$ is the standard hyperquadric $\{\operatorname{Im} w^*=|z^*|^2\}$ and the element $\tau\in\operatorname{Aut}(\mathbb{CP}^2)$ is the affine transformation $(z,w)\longrightarrow(z,+\pi i)$.

As a small consolation for the paradoxical phenomenon, illustrated by Examples 6.2, 6.3, we show now that this does not happen if F is single-valued.

Proposition 6.4. *Let $M\subset\mathbb{C}^n$ be a smooth real-analytic hypersurface, containing a complex hypersurface $X\subset\mathbb{C}^n$ and Levi nondegenerate in $M\setminus X$. Suppose that M is pseudospherical with M^+ being (k,l)-spherical and the multiple-valued analytic mapping F, obtained in Theorem 5.2, is single-valued. Then:*

(i) M^- is also (k,l)-spherical.

(ii) F maps both components M^+, M^- to the same hyperquadric $\mathcal{Q}$.

Proof. Choose $U_1, p^+, p^-, F_0, \mathcal{Q}$ as in the proof of Theorem 6.1 and apply Propositions 3.1 and 4.1 to find a sequence $p_0 = p^+, p_1, ..., p_{2j-1}$ such that the point $p_- \in Q_{p_{2j-1}}$ as well as all the continuations $F_1, ..., F_{2j}$. Since F is single-valued, the continuations are simply restrictions of F onto some domains in $U_1 \setminus X$. By the definition of F_{2j} we have $F_{2j-1}(Q_{p^-}) \subset Q'_{F_{2j}(p^-)}$ so that $F(Q_{p^-}) \subset Q'_{F(p^-)}$. But $p^- \in M$ and accordingly $p^- \in Q_{p^-}$, so $F(p^-) \in Q'_{F(p^-)}$, which means that $F(p^-) \in \mathcal{Q}$. Since $p^- \in M^-$ is arbitrary, this shows that $F(M^-) \subset \mathcal{Q}$ and proves both (i) and (ii). $\square$

Remark 6.5. Theorem 6.1 can be also deduced from the work of Merker [17], [18], where explicit differential equations determining the pseudosphericity of a real hypersurface are presented using a remarkable connection between CR-manifolds and PDE's (see also [26], [25] and also references therein for more details of this connection). The real-analytic nature of the equations implies that if the equations hold for one of the connected components of $M \setminus X$, then they immediately hold for the second connected component. However, the result is not formulated in the above papers. We do that here and give an alternative, complex-analytic proof.

7. The monodromy

In this section we give a description of the multiple-valued extension obtained in Theorem 5.2. It will allow us to find an interesting interaction between nonminimal pseudospherical hypersurfaces in $\mathbb{C}^n$ and linear differential equations of order n.

Let M, X, U_1, p, F_0 satisfy Theorem 5.2, and let F be the (multiple-valued) analytic mapping obtained there. Consider a noncontractible cycle $\gamma : [0,1] \longrightarrow U_1 \setminus X$, $\gamma(0) = \gamma(1) = p$, which is a generator of the fundamental group of $U_1 \setminus X$. Let (F_1, p) be the analytic continuation of the element (F_0, p) of F along γ to the point p. Applying the $\mathcal{Q}$-Segre property of F_0, F_1 and using Proposition 5.1, we obtain a mapping $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$ such that $F_1 = \sigma \circ F_0$. General properties of analytic continuation and the global character of σ imply that the linear automorphism σ is independent:

(i) Of the choice of a generator γ,

(ii) Of the choice of an analytic element $(q, F_{q,0})$ of F at a point $q \in U_1 \setminus X$.

To show (ii), for example, we choose a path γ_q such that $F_{q,0}$ is an extension of F_0 along γ_q and denote by $F_{q,1}$ the extension of $F_{q,0}$ along γ (we suppose, without loss of generality, that $q \in \gamma$). Again, the $\mathcal{Q}$-Segre property of the elements of F and Proposition 5.1 show that there exists an element $\sigma' \in \mathrm{Aut}(\mathbb{CP}^n)$ such that $F_{q,1} = \sigma' \circ F_{q,0}$. Note that $F_{q,1}$ is obviously the extension of F_1 along γ_q. But $F_1 = \sigma \circ F_0$ so that the extension of F_1 along γ_q equals (by the uniqueness) $\sigma \circ F_{q,0}$ and we conclude that $\sigma \circ F_{q,0} = \sigma' \circ F_{q,0}$ and finally $\sigma = \sigma'$, as required. The proof of (i) is analogous.

To see the dependence of σ on the choice of the initial local biholomorphic mapping F_0 of M onto a hyperquadric, choose some other local biholomorphic mapping $\tilde{F}_0$ of (M, p) to a possibly different hyperquadric $\tilde{\mathcal{Q}}$ and denote the continuation of $\tilde{F}_0$ along γ by $\tilde{F}_1$ and the corresponding linear automorphism of $\mathbb{CP}^n$ by $\tilde{\sigma}$. Then applying Proposition 5.1 we conclude that there exists $\tau \in \mathrm{Aut}(\mathbb{CP}^n)$ such that $\tilde{F}_0 = \tau \circ F_0$ and so the continuation of $\tilde{F}_0$ along γ equals $\tau \circ F_1 = \tau \circ \sigma \circ F_0$. On the other hand, $\tilde{F}_1 = \tilde{\sigma} \circ \tilde{F}_0 = \tilde{\sigma} \circ \tau \circ F_0$ so that $\tau \circ \sigma \circ F_0 = \tilde{\sigma} \circ \tau \circ F_0$ and $\tau \circ \sigma = \tilde{\sigma} \circ \tau$.

In fact, the linear automorphism τ is a linear projective equivalence of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. We finally may express $\tilde\sigma$ as follows:

$$\tilde\sigma = \tau \circ \sigma \circ \tau^{-1}. \tag{14}$$

Relation (14) shows that the *monodromy matrix* σ is defined up to matrix conjugation and scaling. We will call this conjugacy class *the monodromy operator of M*. This term is used in analogy with the *monodromy matrix* of a linear differential equation of order n at a singular point [13]. The monodromy operator does not depend on the choice of the cycle γ, the point $q \in U_1 \setminus X$, the element F_q of F, the target hyperquadric $\mathcal{Q}$, and the initial local biholomorphic mapping F_0 of (M,p) to $\mathcal{Q}$ and is only a characteristic of the holomorphic geometry of a nonminimal pseudospherical real-analytic hypersurface. This geometry can be also characterized, for example, by the Jordan normal form of Σ, defined up to scaling of its diagonal part or, alternatively, by the cyclic subgroup $H = \{\sigma^k,\, k \in \mathbb{Z}\} \subset \mathrm{Aut}(\mathbb{CP}^n)$ generated by σ, defined up to conjugation. Note that the subgroup H exactly determines all possible elements of F at a point $q \in U_1 \setminus X$, and all the elements have the form

$$F_{q,k} = \sigma^k \circ F_{q,0}, \quad k \in \mathbb{Z},$$

where $F_{q,0}$ is some fixed element. Both the (scaled) Jordan normal form of Σ, and (the conjugacy class of) the subgroup $H \subset \mathrm{Aut}(\mathbb{CP}^n)$ precisely characterize the monodromy of F about X.

The analogy with differential equations goes even further. Choose the local coordinate system in such a way that X is given in U_1 by the condition $w = 0$. Consider the $(n+1)\times(n+1)$-matrix σ (defined up to scaling). We set

$$A := \frac{1}{2\pi i}\ln\sigma$$

(we may choose any of the matrix logarithms), and consider in a neighbourhood $U(p)$ of p the mapping

$$G_0 : U(p) \longrightarrow \mathbb{CP}^n,\ G_0(z,w) := w^{-A}\cdot F_0(z,w).$$

Here we understand F_0 as the column of its $n+1$ homogeneous components and by w^{-A} we understand the functional matrix exponent $e^{-A\ln w}$. The definition of G_0 does not depend on the choice of the uncertain factor of σ since the uncertain factor clearly just scales the column, representing G_0, and that does not change the element $G_0(z) \in \mathbb{CP}^n$, $z \in U(p)$. Then G_0 extends along an arbitrary path in $U_1 \setminus X$, because F_0 and the matrix-valued mapping w^{-A} do, and determines a (multiple-valued) analytic mapping G. Since the monodromy of w^{-A} is given by

$$w^{-A} \longrightarrow \sigma^{-1}w^{-A} = w^{-A}\sigma^{-1},$$

the monodromy of G is given by

$$G_0 \longrightarrow w^{-A}\cdot\sigma^{-1}\cdot\sigma\cdot F_0 = w^{-A}\cdot F_0 = G_0.$$

Hence, by the Monodromy theorem [21], G is a single-valued holomorphic mapping, and we obtain the following formula characterizing the multiple-valuedness of F:

$$F = w^A \cdot G \qquad \text{(the Monodromy formula)},$$

where G is a holomorphic mapping $U_1 \setminus X \longrightarrow \mathbb{CP}^n$. Note that a very similar formula holds for the monodromy of the fundamental matrix of solutions of a linear differential equation of order n, [13]. The Monodromy formula generalizes Examples 1.1, 5.3, 5.4, and gives a local monodromy representation of an *arbitrary* multiple-valued extension of a local biholomorphic mapping from a nonminimal real hypersurface to a quadric.

We summarize our arguments in the following theorem, which is the expanded formulation of Theorem 2.

Theorem 7.1. *Let $M, X, U_1, \mathcal{Q}$ satisfy Theorem 5.2, and F be the (multiple-valued) analytic mapping obtained there. Then there exists an element $\sigma \in Aut(\mathbb{CP}^n)$ such that:*

1. The monodromy of F with respect to a generator γ of the fundamental group of $U_1 \setminus X$ is given by

$$F_q \longrightarrow \sigma \circ F_q,$$

where F_q is an arbitrary element of F at a point $q \in U_1 \setminus X$. In particular, the collection of all elements of F at a point q is given by the natural action of the cyclic subgroup of $\mathrm{Aut}\,(\mathbb{CP}^n)$, generated by σ, on a fixed analytic element of F at q.

2. All possible changes of the target hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$ and the local biholomorphic map F_0, transferring (M,p) to $\mathcal{Q}$ transform the monodromy matrix σ by the formula

$$\sigma \longrightarrow \tau \circ \sigma \circ \tau^{-1},$$

where $\tau \in Aut(\mathbb{CP}^n)$, and thus generate the monodromy operator Σ. The correspondence

$$M \longrightarrow J(M),$$

where $J(M)$ is the (scaled) Jordan normal form of Σ, is only characterized by holomorphic geometry of M. In particular, $J(M)$ is a biholomorphic invariant of M.

3. If the local coordinates (z,w) at the origin are chosen in such a way that $X = \{w = 0\}$ then there exists a single-valued holomorphic mapping $G : U_1 \setminus X \longrightarrow \mathbb{CP}^n$ such that the following Monodromy formula holds:

$$F = w^A \cdot G,$$

where $2\pi i A$ is a complex logarithm of the monodromy matrix σ.

Remark 7.2. If σ is a scalar matrix, i.e., the monodromy operator Σ is the identity, we conclude, by the Monodromy theorem [21], that the multiple-valued map F is, in fact, a single-valued locally biholomorphic mapping $F : U_1 \setminus X \longrightarrow \mathbb{CP}^n$.

Example 7.3. For the hypersurfaces $M_\alpha \subset \mathbb{C}^2$ given by (12) with $\alpha \in \mathbb{Z}$ and the hypersurfaces $K_m \subset \mathbb{C}^2$ given by (13), the monodromy operator is the identity, and the map F is single-valued. For the hypersurfaces M_α with $\alpha \notin \mathbb{Z}$ the monodromy operator has a diagonal Jordan normal form:

$$J(M_\alpha) = \mathrm{diag}\,\{e^{2\pi i\alpha}, e^{4\pi i\alpha}, 1\}.$$

Thus, the Monodromy representation becomes

$$F = \begin{pmatrix} w^\alpha & 0 & 0 \\ 0 & w^{2\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \\ 1 \end{pmatrix}.$$

Finally, for the model example $M^{\log}$ the (appropriately scaled) Jordan normal form is given by

$$J(M^{\log}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\pi i \\ 0 & 0 & 1 \end{pmatrix}.$$

Decomposing the matrix to the sum of a diagonal and a nilpotent matrices and computing the logarithm, we get $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and so the Monodromy representation takes the form:

$$F = w^A \cdot \begin{pmatrix} z \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \ln w \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z \\ 0 \\ 1 \end{pmatrix}.$$

8. The algebraic case

In this section we show that if the nonminimal pseudospherical hypersurface M in the preimage is algebraic then the multiple-valued extension F, obtained by Theorem 5.2, admits certain holomorphic extension to X.

We start with preliminaries. Let $M \subset \mathbb{C}^n$ be a smooth real-analytic nonminimal hypersurface, containing a complex hypersurface X and Levi nondegenerate in $M \setminus X$. We choose local coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, at the origin in such a way that the complex hypersurface, contained in M, is given by $X = \{w = 0\}$, and M is given locally by the equation

$$\operatorname{Im} w = (\operatorname{Re} w)^m \Phi(z, \bar z, \operatorname{Re} w),$$

where $\Phi(z, \bar z, \operatorname{Re} w)$ is a real-analytic function in a neighbourhood of the origin such that $\Phi(z, \bar z, 0) \neq 0$ identically, $\Phi = O(2)$ and m is a positive integer (see [1] for the existence of such coordinates). We may further consider the local "complex defining equation" (see [1], [18], [10]) of the form

$$w = \bar w\, \Theta(z, \bar z, \bar w),$$

where $\Theta = 1 + O(2)$ is real-analytic. So finally we come to the following defining equation for M:

$$w = \bar w\, e^{i\varphi(z, \bar z, \bar w)}, \tag{15}$$

where the complex-valued real-analytic function φ in a polydisc $U \ni 0$ satisfies the condition $\varphi(z, \bar z, \bar w) = O(2)$ and also the reality condition

$$\varphi(z, \bar z, w\, e^{-i\bar\varphi(\bar z, z, w)}) \equiv \bar\varphi(\bar z, z, w), \tag{16}$$

reflecting the fact that M is a real hypersurface. In what follows we call (13) *the exponential defining equation* for a nonminimal hypersurface M.

Generalizing the ideas in [15], consider in a sufficiently small polydisc $\tilde U \ni 0$ the real-analytic subset

$$\tilde M = \{(z^*, w^*) \in \tilde U : \, w^* = \bar w^* e^{\frac{i}{k}\varphi(z^*, \bar z^*, (\overline{w^*})^k)}\},$$

containing the complex hypersurface $\tilde X = \{w^* = 0\}$, where $k \geq 2$ is an integer. It follows from (16) that $\tilde M$ is in fact a smooth real-analytic hypersurface and that the mapping

$$z^* = z,\; w^* = \sqrt[k]{w}, \quad -\frac{\pi}{k} < \operatorname{Arg} w < \frac{\pi}{k},$$

sends the half of M satisfying $\operatorname{Re} w > 0$ into the half of $\tilde M$ satisfying $\operatorname{Re} w^* > 0$. The hypersurface $\tilde M$ is called *the k-root of M.* Since the inverse mapping

$$\nu : \, z = z^*,\; w = (w^*)^k \tag{17}$$

is holomorphic in all of $\tilde U$ and locally biholomorphic in $\tilde U \setminus \tilde X$, it maps the entire $\tilde M$ into M, preserving the complex hypersurfaces. This means that $\tilde M \setminus \tilde X$ is Levi nondegenerate as well.

Theorem 8.1. *Let M, X, U_1, p, F_0 satisfy Theorem 5.2. Suppose, in addition, that M is real-algebraic and let $F : U_1 \setminus X \longrightarrow \mathbb{CP}^n$ be the (multiple-valued) holomorphic extension obtained in Theorem 5.2. Then:*

(i) *If the mapping F is single-valued then it extends to X holomorphically and $F(M) \subset \mathcal{Q}$.*

(ii) *If the mapping F is multiple-valued then it extends to X as a holomorphic correspondence. Furthermore, if the coordinates (z, w) in U_1 are chosen in such a way that $X \cap U_1 = \{w = 0\}$ then F admits the representation*

$$F(z, w) = \tilde{F}(z, \sqrt[k]{w}),$$

where $\tilde{F} : (\mathbb{C}^n, 0) \longrightarrow \mathbb{CP}^n$ is a single-valued holomorphic mapping, and $k \geq 2$ is an integer.

(iii) *Let $D \subset U \setminus X$, $X \subset \partial D$ be a domain where the multiple-valued mapping F admits a single-valued branch. Then $F|_D$ extends continuously to $D \cup X$.*

Proof. Fix $p \in M$ and F_0 as in Theorem 5.2. By Webster's theorem [27] the graph of the local biholomorphic mapping $F_0 : U(p) \longrightarrow \mathbb{CP}^n$ lies in a complex algebraic set $A \subset U_1 \times \mathbb{CP}^n$ of dimension n. Accordingly, the graph Γ_F of the extended mapping F lies in A as well. Let $\tilde{A}$ be the irreducible component of A containing Γ_F, and let $\pi : \tilde{A} \longrightarrow U_1$ and $\pi' : \tilde{A} \longrightarrow \mathbb{CP}^n$ be the natural projections. Compactness of $\mathbb{CP}^n$ implies that the projection π is proper, so by Remmert's theorem, $\pi(\tilde{A})$ is a complex-analytic subset in U_1, so $\pi(\tilde{A}) = U_1$. Consider now the set

$$E = \{q \in U_1 : \dim(\pi^{-1}(q)) > 0\}.$$

Then E is a complex-analytic subset in U_1 (see, e.g., [16]), and $\dim E < n - 1$ because otherwise $\pi^{-1}(E)$ becomes a complex-analytic subset in $\tilde{A}$ of dimension $\geq n$. Therefore, $X \subsetneq E$ and we can find a point $o \in X$ such that some polydisc $U(o)$ does not contain points from E. To prove (i) we suppose that F is single-valued and choose a projective hyperplane $\Pi \subset \mathbb{CP}^n$ such that Π does not intersect the finite set $\pi'(\pi^{-1}(o)) \subset \mathbb{CP}^n$. Choosing appropriate coordinates in $\mathbb{CP}^n$ we may assume that $\Pi = \mathbb{CP}^n \setminus \mathbb{C}^n$ and accordingly $\pi'(\pi^{-1}(U(o))) \Subset \mathbb{C}^n$. By Riemann's theorem we conclude now that X is a removable singularity for $F|_{U(o)}$. Thus, F extends holomorphically to the complement of E. Since $\dim E \leq n - 2$, it follows that F extends holomorphically to all of U (see, e.g., [11]). The inclusion $F(M) \subset \mathcal{Q}$ follows from the uniqueness.

To prove (ii) we note that, by algebraicity of F_0, its multiple-valued extension F is in fact finite-valued which means that there exists an integer $k \geq 2$ such that the extension of the analytic element (F_0, p) along the path γ^k, where γ is the generator of $\pi_1(U_1 \setminus X)$, does not change this element. Choose now the coordinates (z, w) in U_1 in such a way that $X = \{w = 0\}$ and $p \in M^+$ and consider $\tilde{M}$ - the k-root of M. It follows from the arguments above that $\tilde{M}$ satisfies all the conditions for Theorem 5.2 and we may also consider the multiple-valued mapping $\tilde{F}$, corresponding to the pseudospherical hypersurface $\tilde{M}$. The map ν given by (17) gives the relation between M and $\tilde{M}$, and thus shows that the monodromy of $\tilde{F}$ with respect to the generator γ of $\pi_1(\tilde{U} \setminus X)$ is simply the identity, since the monodromy of F with respect to γ^k is the identity. We conclude that the map $\tilde{F}$ is single-valued and from claim (i), $\tilde{F}$ extends to $\tilde{X}$ holomorphically, and the explicit formula for ν now implies (ii).

For the proof of (iii) (only the multiple-valued case is not immediate) it is easy to see from (ii) that for each $o = (z_0, 0) \in X$ the limit $\lim\limits_{(z,w)\to o} F|_D(z, w) = \tilde{F}(z_0, 0)$, which shows the continuity of the glued map in $D \cup X$. This completes the proof of the theorem.

□

Remark 8.2. In the case when $\mathcal{Q}$ is strictly pseudoconvex and F is single valued, the set $F(X)$ in the above theorem becomes a connected locally analytic set in $\mathcal{Q}$ so that $F(X)$ consists of a unique point. Using the k-root construction it is easy to verify from here that in the multiple-valued case the cluster set of X with respect to any single-valued branch of F in a domain $D \subset U \setminus X$, $X \subset \partial D$ consists of exactly one point.

9. Application to automorphisms: towards the Dimension Conjecture

The characterization of the monodromy of the multiple-valued mapping F, obtained in Section 7, allows us to prove a useful fact on the structure of the local automorphism group of a nonminimal pseudospherical hypersurface.

Let $M, X, p, U_1, F_0, F, \mathcal{Q}$ be as before. Without loss of generality we assume $p \in M^+$. We denote by $\mathrm{Aut}\,(M,0)$ the local automorphism (pseudo)group of the smooth real-analytic germ $(M,0)$. Note that we consider only the connected component of the identity of the local automorphism (pseudo)group, so that the automorphisms preserve the sides M^+, M^-. We also assume F_0 to be locally biholomorphically extended along M^+. If p is sufficiently close to 0, then each automorphism $\psi \in \mathrm{Aut}\,(M,0)$, close to the identity, is defined in a neighbourhood of p and we may consider its push-forward

$$\tau := F_0 \circ \psi \circ F_0^{-1} \in \mathrm{Aut}\,(\mathcal{Q}, p'),$$

where $p' = F_0(p)$. Then, by [5], $\tau \in \mathrm{Aut}\,(\mathcal{Q})$, the correspondence $\psi \to \tau$ is injective, and we obtain an embedding of $\mathrm{Aut}\,(M,0)$ into $\mathrm{Aut}\,(\mathcal{Q})$ by sending ψ to τ . We denote the image by G_M. The subgroup $G_M \subset \mathrm{Aut}\,(\mathcal{Q})$ can be naturally identified with $(\mathrm{Aut}\, M, 0)$. Now let us consider the analytic mapping $F_\psi := F \circ \psi$ in $U \setminus X$ with a sufficiently small polydisc $U \subset U_1$, centred at 0 (note that the local automorphism ψ must preserve the complex hypersurface X). Then it is easy to see from the definition of F_ψ that its germ at p also maps (M,p) to $\mathcal{Q}$, and if σ is the monodromy matrix of F then F_ψ has the same monodromy matrix σ. On the other hand, relation (12) shows that the monodromy of F_ψ is given by the matrix $\tau \circ \sigma \circ \tau^{-1}$ with τ being exactly the push-forward of ψ (as it follows from the definition of τ). We get the identity

$$\sigma = \tau \circ \sigma \circ \tau^{-1}$$

showing that the push-forwarded automorphism group G_M is a subgroup of the group $C \subset \mathrm{Aut}\,\mathcal{Q}$, which is the intersection of the centralizer $Z(\sigma)$ of the element $\sigma \in \mathrm{Aut}\,(\mathbb{CP}^n)$ with the subgroup $\mathrm{Aut}\,(\mathcal{Q}) \subset \mathrm{Aut}\,(\mathbb{CP}^n)$. This yields the claim of Theorem 3, which we reformulate here in the following way.

Theorem 9.1. *Let $M \subset \mathbb{C}^n$ be a smooth real-analytic hypersurface, containing a complex hypersurface $X \subset \mathbb{C}^n$ and Levi nondegenerate in $M \setminus X$. Suppose that $0 \in X$, M is pseudospherical, $F_0 : (M,p) \longrightarrow \mathcal{Q}$ is a local biholomorphic mapping to a nondegenerate hyperquadric $\mathcal{Q} \subset \mathbb{CP}^n$, F is the associated analytic mapping, and σ is the monodromy matrix of M. Then the local automorphism (pseudo)group $Aut(M,0)$ can be naturally embedded to the subgroup $C = Z(\sigma) \cap Aut(\mathcal{Q}) \subset Aut(\mathbb{CP}^n)$, where $Z(\sigma)$ is the centralizer of the element σ in $Aut(\mathbb{CP}^n)$.*

As an application of the representation, obtained in Theorem 9.1, we discuss now the so-called *Dimension Conjecture* formulated in the survey [3]. Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, $0 \in M$. Consider the local automorphism (pseudo)group $G = \mathrm{Aut}\,(M,0)$ and also the stability group $\mathrm{Aut}_0\,(M,0)$ of $(M,0)$, which we denote by H. When M is Levi-flat,

$\dim G = \dim H = \infty$. If M is Levi nondegenerate at 0, then the classical results in [20], [5] imply that

$$\dim G \leq 8,\ \dim H \leq 5.$$

Further analysis in [2] shows $\dim H \leq 1$ in the case when the germ $(M, 0)$ is not spherical. If M is Levi-degenerate at 0, but not Levi-flat, simple arguments in [3] give the estimate $\dim G \leq 8$, but nothing for $\dim H$, except the trivial corollary $\dim H \leq 8$.

In the Levi-degenerate and non-Levi-flat case, the hypersurface M can either be of finite type at 0 (see [1] for various possible type conditions) which is equivalent to its minimality, or M can be of infinite type (which is equivalent to its nonminimality). Some generalizations of Poincare-Chern-Moser arguments can be used to provide the estimate $\dim H \leq 3$ in the finite type case (see, for example, [14]). But the infinite type case turns out to be quite difficult, since no generalizations of the Poincare-Chern-Moser arguments are possible in this case. In some particular cases the estimate $\dim H \leq 5 = \dim \mathrm{Aut}_p(S^3)$ was obtained in [4], [15] (here $\mathrm{Aut}_p(S^3)$ denotes the stability group of a point $p \in S^3$). However, no general approach to analyse local automorphisms in the infinite type case is known so far. These difficulties motivate the following open question.

The Dimension Conjecture [3]**:** Let $(M, 0) \subset \mathbb{C}^2$ be a smooth real-analytic germ and suppose that M is not Levi-flat. Then the following bound holds:

$$\dim \mathrm{Aut}_0\,(M, 0) \leq \dim \mathrm{Aut}_p(S^3) = 5.$$

From the above discussion it follows that only the nonminimal case remains open. Moreover, suppose that $\dim \mathrm{Aut}_0\,(M, 0) \geq 5$. Since the subset of points in M where M is Levi-degenerate is nowhere dense, we can find, in a neighbourhood of 0, a point $p \in M$ where M is Levi nondegenerate. Then the results of [2] applied to (M, p) imply that the germ (M, p) is spherical, i.e., M must be nonminimal and spherical at a generic point. To satisfy all the conditions required by Theorems 5.2, 9.1, we note that if any neighbourhood of 0 contains a Levi degenerate point p of finite type, then the estimates from [14] applied to (M, p) give

$$\dim \mathrm{Aut}_0\,(M, 0) \leq \dim \mathrm{Aut}\,(M, p) \leq 4,$$

which is a contradiction. Hence, we conclude that *there exists a neighbourhood U of the origin such that $(M \setminus X) \cap U$ is Levi nondegenerate.* Applying Theorems 5.2, 9.1 we finally obtain the estimate:

$$\dim \mathrm{Aut}\,(M, 0) \leq \dim\,(Z(\sigma) \cap \mathrm{Aut}\,(\mathcal{Q})).$$

Centralizers of elements $\sigma \in \mathrm{GL}_3(\mathbb{C})$ are well known: the centralizer of a scalar matrix is the entire $\mathrm{GL}_3(\mathbb{C})$ and has dimension 9, for any other element the dimension of the centralizer is at most 5. Taking scaling into account, we get the following corollary.

Corollary 9.2. *Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, containing a complex hypersurface $X \subset \mathbb{C}^2$ and Levi nondegenerate in $M \setminus X$. Suppose that $0 \in X$, M is spherical, $F_0 : (M, p) \longrightarrow S^3$ is a local biholomorphic mapping, F is the associated analytic mapping and the monodromy operator Σ of F is not identical. Then*

$$\dim \mathrm{Aut}\,(M, 0) \leq 4.$$

Also note that if the multiple-valued mapping F is, in fact, single-valued and extends holomorphically to X then, by uniqueness, $F(X)$ is a connected locally complex-analytic set, lying in S^3, which means that $F(X)$ is a unique point $p \in S^3$ and the push-forward of $(\mathrm{Aut}\, M, 0)$ becomes a subgroup in the 5-dimensional stability group $\mathrm{Aut}_p(S^3)$. Collecting all the arguments above and applying Proposition 6.4 and Theorem 8.1 yields

Corollary 9.3 (towards the Dimension Conjecture). *The Dimension Conjecture holds for any smooth real-analytic nonflat germ $(M,0)\subset\mathbb{C}^2$ except, possibly, a germ of a hypersurface M, containing a complex hypersurface $X\ni 0$, Levi nondegenerate in $M\setminus X$ and admitting a single-valued locally biholomorphic mapping $F: U\setminus X\longrightarrow\mathbb{CP}^2$ such that $F(M\setminus X)\subset S^3$. Furthermore, one can assume that M is nonalgebraic and F does not extend holomorphically to X. Here U is some neighbourhood of the origin.*

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